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A generalized view on Galilean invariance in stabilized compressible flow computations

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SUMMARY

The present work presents a generalized interpretation and further elaborations on the significance of Galilean invariance in compressible flow computations with stabilized and variational multi-scale methods. In the present work, the use of a matrix-operator description of Galilean transformations provides an improved setting for the understanding of the key issues, and the development of more general approaches to Galilean-invariant stabilization. Copyright © 2007 John Wiley & Sons, Ltd.

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1. Introduction

The present work generalizes the discussion on the Galilean invariance properties that well-posed stabilization operators need to satisfy [15, 16, 17, 18]. Lagrangian compressible flow computations in [17] showed that non-invariant SUPG formulations may lead to unstable patterns in the solution, as shown in Figure 1. The Galilean-invariance analysis presented here is carried out casting Galilean transformations as a matrix group acting on the vector flow equations. This idea was also used in [20], although the main results of the analysis in the present work contrast with the arguments proposed therein.

The key point in the subsequent discussion can be summarized very simply. Let $\boldsymbol{V}^G$ be a constant velocity field representing the Galilean *velocity shift*. The original space-time reference

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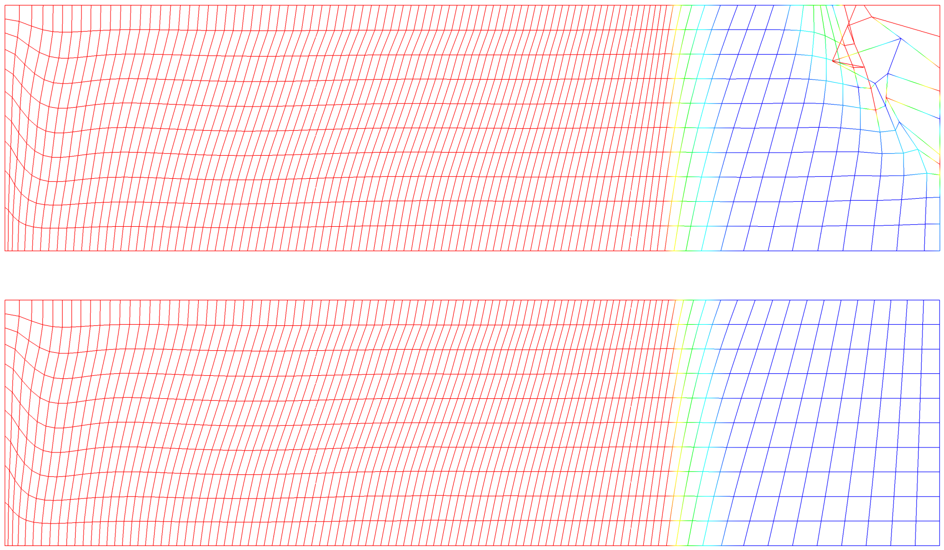

Figure 1. Results from the computations in [17]. Mesh distortion plot: The color scheme represents the pressure. Above: SUPG formulation violating Galilean invariance. Below: SUPG abiding the Galilean invariance principle. A classical quadrilateral Saltzmann mesh is used in an implosion computation. The initial velocity is of unit magnitude and directed horizontally from right to left, except the left boundary which is held fixed. The initial density is unity and the initial specific internal energy is 10^{-1}. A shock forms at the left boundary and advances to the right. Note the *mesh coasting* phenomenon on the top right corner of the upper domain, absent in the SUPG formulation satisfying Galilean invariance, below. Note also that mesh tangling occurs ahead of the shock front, in a region where flow should undergo a simple translation. More details on this computation can be found in [17].

frame $[\boldsymbol{x}, t]$ and the transformed reference frame $[\tilde{t}, \tilde{\boldsymbol{x}}]$ are related by the following mapping:

$$\boldsymbol{x} = \tilde{\boldsymbol{x}} + \boldsymbol{V}^G \, \tilde{t} \, , \tag{1}$$

$$t = \tilde{t} \, . \tag{2}$$

Now consider a typical shape function ψ^h used to perform the Bubnov-Galerkin projection (for the moment, imagine that no stabilizing operator of SUPG type is applied in the variational form). Clearly, when the change of reference frames is performed, ψ^h transforms as:

$$\psi^h(\boldsymbol{x}, t) = \psi^h(\tilde{\boldsymbol{x}} + \boldsymbol{V}^G \tilde{t}, \tilde{t}) \, , \tag{3}$$

that is, the spatial coordinates at which ψ^h is centered are shifted by $\boldsymbol{V}^G t$. It will be shown that, as a consequence, *Galerkin orthogonality* between the residual and the test space is preserved under a change of observer.

SUPG and variational multiscale stabilized methods [2, 6, 7, 8, 9, 10, 11, 12] are Petrov-Galerkin methods in which the *local* structure of the partial differential equations is used to perturb the Bubnov-Galerkin test space. In this sense, these methods are *locally/physically* adapted Petrov-Galerkin methods, aimed to improve the overall stability properties of the underlying Bubnov-Galerkin formulation. If we indicate by p^h the perturbation to the test function space introduced by the SUPG method, also p^h should satisfy relationship (3). If this is not the case, the effect of the SUPG operators is not frame-invariant, with potentially negative consequences on the overall stability of the method, as shown in the numerical tests in [17]. This work is focussed on the case of compressible fluids, which present a more challenging analysis. As already pointed out in [15, 16] the case of incompressible fluids is somewhat simpler to analyze, and SUPG operators developed using the quasi-linear advective form of the Navier-Stokes or Euler equations are automatically guaranteed to be Galilean invariant.

The fundamental reason why extra care has to be taken in developing frame-invariant SUPG operators for compressible flows is that the SUPG methodology is based on a local linearization of the nonlinear equations of gas dynamics. As a consequence, quadratic and higher-order terms are neglected in the construction of the test function perturbation. This fact, in turn, prevents SUPG formulations from being automatically frame invariant, unless special attention is paid to ensuring this property.

Because, in SUPG approaches, p^h is a function of the gradient of the shape function ψ^h, global conservation is not prevented by non-invariant SUPG operators: *Invariance with respect to Galilean transformations is a local concept.* Therefore, numerical tests evaluating invariance of global total energy budgets are not meaningful assessments of (local) invariance properties of stabilizing operators.

The rest of the exposition proceeds as follows: Using the approach already explored in [15, 16], an arbitrary Lagrangian-Eulerian formulation is presented in Section 2, its stabilized variational formulation is developed, and, finally, the matrix approach to invariance analysis is applied in Section 3, to establish criteria for and examples of invariant SUPG operators. Conclusions are summarized in Section 4.

2. Stabilized ALE equations of compressible flows

In [15, 16], the arbitrary Lagrangian-Eulerian (ALE) approach was used to develop a general analysis of Galilean invariance. In [15] the ALE equations were developed in a *referential* frame, associated with the initial configuration of the computational mesh, while in [16] the equations were written in the more intuitive *current configuration* reference frame. In the present work, the latter approach is followed.

2.1. Kinematics

Let the open sets $\hat{\Omega}$ and Ω in $\mathbb{R}^{n_d}$ be the domains occupied by the mesh in its initial and current configurations, respectively (see, Fig. 2). The *mesh deformation* $\hat{\boldsymbol{\varphi}}$ is the transformation from the original to the current mesh configuration:

$$\hat{\boldsymbol{\varphi}} : \hat{\Omega} \to \Omega = \hat{\boldsymbol{\varphi}}(\hat{\Omega}) \ , \tag{4}$$

$$\boldsymbol{\chi} \mapsto \boldsymbol{x} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t), \quad \forall \boldsymbol{\chi} \in \hat{\Omega},\ t \geq 0 \ , \tag{5}$$

where, according to the previous definitions, $\boldsymbol{\chi} = \boldsymbol{x}(t=0)$. Mesh displacements/velocities are defined as

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, 0) = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \boldsymbol{\chi} \ , \tag{6}$$

$$\hat{\boldsymbol{v}} = \partial_t|_{\boldsymbol{\chi}} \hat{\boldsymbol{\varphi}} \tag{7}$$

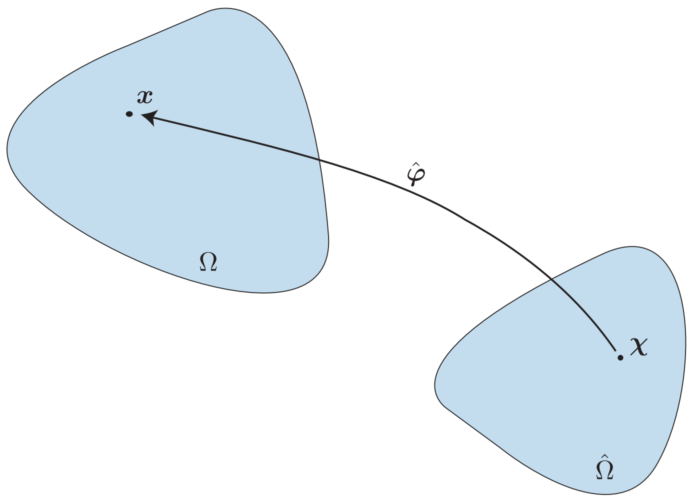

Figure 2. Sketch of the map $\hat{\boldsymbol{\varphi}}$ for the generalized ALE framework.

The *mesh deformation gradient* and the *mesh Jacobian determinant* are defined as

$$\hat{\boldsymbol{F}} = \nabla_{\boldsymbol{\chi}} \hat{\boldsymbol{\varphi}} \ , \tag{8}$$

$$\hat{J} = \det \hat{\boldsymbol{F}} \ , \tag{9}$$

where $\nabla_{\boldsymbol{\chi}}$ is the gradient in the referential frame. The chain rule and a number of calculus manipulations [15, 16], yield an expression for the Lagrangian time derivative of a scalar-valued function f:

$$\dot{f}(\boldsymbol{\chi}, t) = \partial_t|_{\boldsymbol{\chi}} f + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} f \ , \tag{10}$$

where $\nabla_{\boldsymbol{x}}$ is the gradient in the current configuration, and $\boldsymbol{c} = \boldsymbol{v} - \hat{\boldsymbol{v}}$ is the *convective velocity*, or relative velocity of the material with respect to the mesh. In the Lagrangian limit, $\hat{\boldsymbol{v}} = \boldsymbol{v}$ and $\boldsymbol{c} = \boldsymbol{0}$. In the Eulerian limit, $\hat{\boldsymbol{v}} = \boldsymbol{0}$ and $\boldsymbol{c} = \boldsymbol{v}$.

2.2. Equations in conservative vector form

In [16], the Leibniz transport theorem is used to derive the integral ALE equations for a control volume Ω whose boundaries move with an arbitrary velocity $\hat{\boldsymbol{v}}$:

$$0 = \int_\Omega \hat{J}^{-1} \, \partial_t|_{\boldsymbol{\chi}} (\hat{J}\rho) \; d\Omega + \int_\Gamma \rho \; \boldsymbol{c} \cdot \boldsymbol{n} \; d\Gamma \ , \tag{11}$$

$$0 = \int_\Omega \hat{J}^{-1} \, \partial_t|_{\boldsymbol{\chi}} (\hat{J}\rho \boldsymbol{v}) \; d\Omega + \int_\Gamma (\rho \; \boldsymbol{v} \otimes \boldsymbol{c} - \boldsymbol{\sigma}) \boldsymbol{n} \; d\Gamma - \int_\Omega \rho \; \boldsymbol{g} \; d\Omega \ , \tag{12}$$

$$0 = \int_\Omega \hat{J}^{-1} \, \partial_t|_{\boldsymbol{\chi}} (\hat{J}\rho E) \; d\Omega + \int_\Gamma (\rho \, E \, \boldsymbol{c} - \boldsymbol{\sigma}^T \boldsymbol{v} + \boldsymbol{q}) \cdot \boldsymbol{n} \; d\Gamma - \int_\Omega \rho \; (\boldsymbol{v} \cdot \boldsymbol{g} + s) \; d\Omega \ . \tag{13}$$

Here E is the total energy, the sum of of the internal energy e and the kinetic energy $\boldsymbol{v} \cdot \boldsymbol{v}/2$, $\boldsymbol{g}$ the body force, and s the heat source. All previous quantities are defined per unit mass. In addition, ρ is the density, $\boldsymbol{\sigma}$ is the stress tensor, and $\boldsymbol{q}$ is the heat flux, which, for convenience, will be omitted in the following derivations. Applying the divergence theorem in its vector and tensor forms, the following conservative vector form of (11)–(13) can be derived:

$$\boxed{\mathsf{Res}_{\mathsf{U}}(\boldsymbol{Y}) = \hat{J}^{-1}\ \partial_t|_{\boldsymbol{\chi}}(\hat{J}\ \boldsymbol{U}(\boldsymbol{Y})) + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{G}(\boldsymbol{Y}) + \boldsymbol{Z}(\boldsymbol{Y}) = \boldsymbol{0}\ ,} \tag{14}$$

where $\boldsymbol{G}$ is a $(n_d + 2) \times n_d$-matrix, and its divergence is taken over the second index, that is $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{G} = \partial_{x_i}\boldsymbol{G}_i$, with $\boldsymbol{G}_i$ the ith column of $\boldsymbol{G}$. The vector $\boldsymbol{Y}$ indicates the set of solution variables which are directly solved in the computations. $\boldsymbol{Y}$ may or may not be equal to the vector of conserved quantities $\boldsymbol{U}$. Assuming that the stress is given by a simple isotropic pressure, $\boldsymbol{\sigma} = -p\boldsymbol{I}_{n_d\times n_d}$, and, without lack of generality, that the heat flux is absent (i.e., $\boldsymbol{q} = \boldsymbol{0}$), the vectors in (14) can be expressed as

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ \rho\boldsymbol{v} \\ \rho E \end{bmatrix}, \quad \boldsymbol{Z} = -\begin{bmatrix} 0 \\ \rho\boldsymbol{g} \\ \rho\boldsymbol{v}\cdot\boldsymbol{g} + \rho s \end{bmatrix}, \tag{15}$$

$$\boldsymbol{G} = \boldsymbol{U}\boldsymbol{c}^T + \boldsymbol{G}^L, \quad \boldsymbol{U}\boldsymbol{c}^T = \begin{bmatrix} \rho\boldsymbol{c}^T \\ \rho\boldsymbol{v}\otimes\boldsymbol{c} \\ \rho E\boldsymbol{c}^T \end{bmatrix}, \quad \boldsymbol{G}^L = \begin{bmatrix} \boldsymbol{0}_{1\times n_d} \\ p\boldsymbol{I}_{n_d\times n_d} \\ p\boldsymbol{v}^T \end{bmatrix}, \tag{16}$$

where the tensor product $\boldsymbol{v}\otimes\boldsymbol{c}$ can also be written as $\boldsymbol{v}\boldsymbol{c}^T$. Compressible fluids abide an equation of state for the pressure of the form $p = p(\rho, e)$, which can be used to derive a quasi-linear form of (14). Following derivations analogous to [16],

$$\begin{aligned} \mathsf{Res}_{\mathsf{U}}(\boldsymbol{Y}) &= \hat{J}^{-1}\partial_t|_{\boldsymbol{\chi}}(\hat{J}\boldsymbol{U}) + \partial_{x_i}\boldsymbol{G}_i + \boldsymbol{Z} \\ &= \partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \boldsymbol{U}\,\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \partial_{x_i}(c_i\boldsymbol{U} + \boldsymbol{G}_i^L) + \boldsymbol{Z} \\ &= \partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \boldsymbol{U}\,\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \boldsymbol{U}\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{c} + c_i\partial_{x_i}\boldsymbol{U} + \partial_{x_i}\boldsymbol{G}_i^L + \boldsymbol{Z} \\ &= \partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + c_i\partial_{x_i}\boldsymbol{U} + \boldsymbol{U}\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} + \partial_{x_i}\boldsymbol{G}_i^L + \boldsymbol{Z}\ . \end{aligned} \tag{17}$$

Let us now define:

$$\boldsymbol{A}_0 = \frac{\partial\boldsymbol{U}}{\partial\boldsymbol{Y}}\ , \tag{18}$$

$$\boldsymbol{A}_i^{L;I}\partial_{x_i}\boldsymbol{Y} = \boldsymbol{U}\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}\ , \tag{19}$$

$$\boldsymbol{A}_i^{L;II} = \frac{\partial\boldsymbol{G}_i^L}{\partial\boldsymbol{Y}}\ , \tag{20}$$

$$\boldsymbol{C} = \frac{\partial\boldsymbol{Z}}{\partial\boldsymbol{Y}}\ . \tag{21}$$

Hence,

$$\mathsf{Res}_{\mathsf{U}}(\boldsymbol{Y}) = \boldsymbol{A}_0\partial_t|_{\boldsymbol{\chi}}\boldsymbol{Y} + \left(c_i\boldsymbol{A}_0 + \boldsymbol{A}_i^{L;I} + \boldsymbol{A}_i^{L;II}\right)\partial_{x_i}\boldsymbol{Y} + \boldsymbol{C}\boldsymbol{Y}\ . \tag{22}$$

Denoting $\boldsymbol{A}_i = c_i\boldsymbol{A}_0\partial_{x_i} + \boldsymbol{A}_i^{L;I} + \boldsymbol{A}_i^{L;II}$, (17) yields

$$\boxed{\mathsf{Res}_{\mathsf{U}}(\boldsymbol{Y}) = \boldsymbol{A}_0\ \partial_t|_{\boldsymbol{\chi}}\boldsymbol{Y} + \boldsymbol{A}_i(\boldsymbol{Y})\ \partial_{x_i}\boldsymbol{Y} + \boldsymbol{C}(\boldsymbol{Y})\ \boldsymbol{Y} = \boldsymbol{0}\ .} \tag{23}$$

The specific form of the matrices $\boldsymbol{A}_0$, $\boldsymbol{A}_i$'s, and $\boldsymbol{C}$ depends on the choice of variables in the solution vector $\boldsymbol{Y}$.

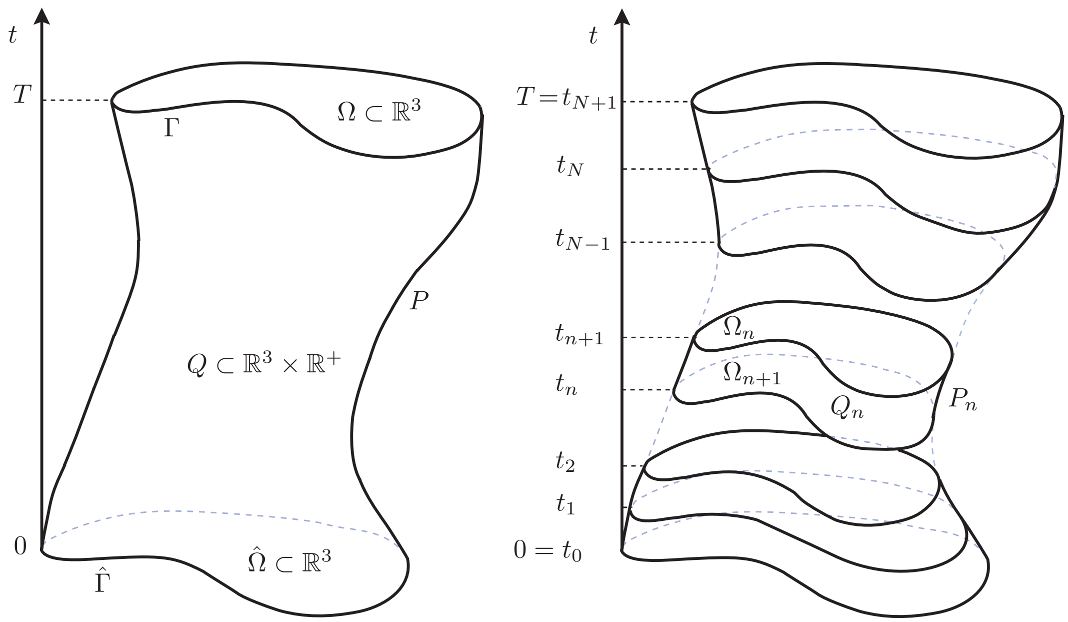

Figure 3. Space-time domain (left) and slicing into space-time slabs (right).

2.3. A space-time variational formulation

A space-time variational formulation is now derived from (14), to develop the analysis of Galilean invariance. Similar conclusions in the context of Galilean invariance analysis hold for alternative space-time or semi-discrete formulations. Given a partition $0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T$ of the time interval $I =]0, T]$, let $I_n =]t_n, t_{n+1}]$, so that $]0, T] = \bigcup_{n=0}^{N-1} I_n$. The space-time domain $\hat{Q} = \hat{\Omega} \times I$ can be divided into time slabs $\hat{Q}_n = \hat{\Omega} \times I_n$, with "lateral" boundary $\hat{P}_n = \hat{\Gamma} \times I_n$ ($\hat{\Gamma} = \partial\hat{\Omega}$ is the boundary of $\hat{\Omega}$, see Fig. 3). The domain $\hat{\Omega}$ is divided into element subdomains $\hat{\Omega}^e$, so that $\hat{\Omega} = \overline{\bigcup_{e=1}^{n_{el}} \hat{\Omega}^e}$, and a typical space-time element is given by tensor product $\hat{Q}_n^e = \hat{\Omega}^e \times I_n$. Hence, the space-time discretization adopted is *prismatic* in time, with respect to the referential frame. Analogously, the space-time boundary is partitioned as $\hat{P}_n = \hat{P}_n^g \cup \hat{P}_n^h$, $\hat{P}_n^g \cap \hat{P}_n^h = \emptyset$, where $\hat{P}_n^g$ and $\hat{P}_n^h$ are the space-time Dirichlet and Neumann boundary, respectively. The previous definitions can be *pushed forward* to the current configuration (see Fig. 3):

$$Q_n \stackrel{\text{def}}{=} \Omega \times I_n = \hat{\boldsymbol{\varphi}}(\hat{Q}_n) \,, \tag{24}$$

$$P_n \stackrel{\text{def}}{=} \hat{\boldsymbol{\varphi}}(\hat{P}_n) \,, \tag{25}$$

$$\Gamma_n \stackrel{\text{def}}{=} \hat{\boldsymbol{\varphi}}(\hat{\Gamma}, t_n) \,, \tag{26}$$

and, analogously,

$$\Omega_n^e \stackrel{\text{def}}{=} \hat{\boldsymbol{\varphi}}(\hat{\Omega}^e, t_n) , \tag{27}$$

$$Q_n^e \stackrel{\text{def}}{=} \Omega^e \times I_n = \hat{\boldsymbol{\varphi}}(\hat{Q}_n^e) , \tag{28}$$

$$P_n^e \stackrel{\text{def}}{=} \hat{\boldsymbol{\varphi}}(\hat{P}_n^e) . \tag{29}$$

Note, in particular, that $\Omega_n \times I_n \neq \Omega \times I_n$, and $\Omega_n^e \times I_n \neq \Omega^e \times I_n$, due to the structure of the mapping $\hat{\boldsymbol{\varphi}}$. The variational statement reads:

Find $\boldsymbol{Y}^h \in \mathcal{S}_n^h$, such that, $\forall \boldsymbol{W}^h \in \mathcal{V}_n^h$,

$$\mathsf{B}(\boldsymbol{W}^h, \boldsymbol{Y}^h) + \mathsf{SUPG}(\boldsymbol{P}^h(\boldsymbol{W}^h), \boldsymbol{Y}^h) + \mathsf{DC}(\boldsymbol{W}^h, \boldsymbol{Y}^h) = \mathsf{F}(\boldsymbol{W}^h) , \tag{30}$$

with

$$\begin{aligned} \mathsf{B}(\boldsymbol{W}^h, \boldsymbol{Y}^h) = & \int_{\Omega(t_{n+1})} \boldsymbol{W}^h(\boldsymbol{\chi}, t_{n+1}^-) \cdot \boldsymbol{U}(\boldsymbol{Y}^h(\boldsymbol{x}, t_{n+1}^-)) \; d\Omega \\ & - \int_{\Omega(t_n)} \boldsymbol{W}^h(\boldsymbol{\chi}, t_n^+) \cdot \boldsymbol{U}(\boldsymbol{Y}^h(\boldsymbol{x}, t_n^-)) \; d\Omega \\ & - \int_{Q_n} \Big(\boldsymbol{W}^h_{,t} \cdot \boldsymbol{U}_i(\boldsymbol{Y}^h) + \boldsymbol{W}^h_{,i} \cdot \boldsymbol{G}_i(\boldsymbol{Y}^h) \Big) \; dQ \\ & + \int_{Q_n} \boldsymbol{W}^h \cdot \boldsymbol{Z}(\boldsymbol{Y}^h) \; dQ + \int_{P_n^g} \boldsymbol{W}^h \cdot \boldsymbol{G}_i(\boldsymbol{Y}^h) n_i dP , \end{aligned} \tag{31}$$

$$\mathsf{F}(\boldsymbol{W}^h) = - \int_{P_n^h} \boldsymbol{W}^h \cdot \boldsymbol{H}_i n_i \; dP , \tag{32}$$

The symbol n_i stands for the ith component of the normal to the space-time boundary, and $\boldsymbol{H}_i$ is the Neumann flux across the boundary in the ith direction. Hence Neumann boundary conditions are incorporated as *natural* boundary conditions in the variational form, while Dirichlet boundary conditions are *strongly* embedded in the definition of the function spaces utilized. In the analysis that follows it is not necessary to specifically define the spatial discretization utilized in practical computations. For example, piece-wise linear iso-parametric finite elements in space can be used, but the analysis applies to a larger class of function spaces, with higher polynomial order and/or regularity.

Note that the first two integrals in the definition of $\mathsf{B}(\cdot, \cdot)$ *weakly* impose continuity of the solution between time-slabs. This can be verified by performing integration by parts in time and space, to recover the underlying Euler-Lagrange equations. The proposed variational statement encompasses a large class of time-integration schemes. If equal-order, discontinuous-in-time interpolation of order p is used, one obtains a family of time integrators of order $2p+1$ [5, 19]. If, instead, continuous interpolation of order p is used for the trial space in time and discontinuous interpolation of order $p-1$ for the test space, one obtains a family of time-integrators of order $2p$ [1, 3, 4, 13, 14]. The member of the latter family for $p = 1$ was used in shock hydrodynamics computations in [17, 18]. Finally, by evaluating the space-time integrals using approximate quadratures in time, it is also possible to recover other semi-discrete time integrators, such as the mid-point or Crack-Nicholson methods. It can also be easily verified that the proposed formulation embeds global conservation of mass, momentum and total energy.

The term $\mathsf{DC}(\boldsymbol{W}^h, \boldsymbol{Y}^h)$ is a discontinuity capturing operator which may be present in compressible flow computations to control oscillations of the solution near shocks. Away from discontinuities, well-posed discontinuity capturing operators are usually negligible in magnitude. In the analysis that follows, without loss of generality, $\mathsf{DC}(\boldsymbol{W}^h, \boldsymbol{Y}^h)$ is assumed to vanish. SUPG stabilization operators take the form

$$\mathsf{SUPG}(\boldsymbol{P}^h(\boldsymbol{W}^h), \boldsymbol{Y}^h) = -\sum_{e=1}^{(n_{el})_n} \int_{Q_n^e} \boldsymbol{P}^h(\boldsymbol{W}^h) \cdot \mathbf{Res}_{\mathbf{U}}(\boldsymbol{Y}^h)\; dQ\,. \tag{33}$$

It is not necessary to specify a detailed expression for $\boldsymbol{P}^h$ at this point in the discussion. The presentation of the main ideas will proceed at a much more general level. A specific form of $\boldsymbol{P}^h$ will be given in Section 3.4, when detailed aspects of the design of SUPG operators will be assessed from the point of view of frame invariance.

3. Galilean transformations

Galilean transformations are commonly used to verify the consistency of physical and computational models. A well-designed, consistent model must be Galilean invariant, or, more precisely, its functional form $\mathcal{M}$ has to transform as

$$\mathcal{M}(\boldsymbol{v}, \boldsymbol{x}, t, \dots) \xrightarrow{\mathsf{G}} \mathcal{M}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{x}}, \tilde{t}, \dots)\,, \tag{34}$$

meaning that the equations of the system dynamics must be written in the same way in two reference frames which differ by a Galilean transformation. A finite element method is generally developed over a geometrical model, by means of the computational grid or *mesh*, a discrete subdivision of the physical space. The case of a fixed (Eulerian), and arbitrary moving (ALE) mesh are considered next.

3.1. The Galilean shift in different reference frames

3.1.1. Current configuration (Eulerian) reference frame A Galilean transformation consists of a *shift* in the spatial coordinate by $\boldsymbol{V}^G t$, and is most commonly and intuitively described by the *affine* mapping in the case of the current configuration reference frame:

$$\mathsf{G} : \mathbb{R}^+ \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \longrightarrow \mathbb{R}^+ \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d}, \tag{35}$$

$$\left[t\; \boldsymbol{x}^T\; \boldsymbol{v}^T\right]^T \mapsto \left[\tilde{t}\; \tilde{\boldsymbol{x}}^T\; \tilde{\boldsymbol{v}}^T\right]^T, \tag{36}$$

$$\begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & \mathbf{0}_{1\times n_d} \\ -\boldsymbol{V}^G & \boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times n_d} \\ \mathbf{0}_{n_d\times 1} & \mathbf{0}_{n_d\times n_d} & \boldsymbol{I}_{n_d\times n_d} \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{0}_{n_d\times 1} \\ \boldsymbol{V}^G \end{bmatrix}. \tag{37}$$

Consequently, the following transformation applies to the temporal and spatial derivatives:

$$\partial_t|_{\boldsymbol{x}} = \partial_{\tilde{t}}|_{\tilde{\boldsymbol{x}}} - \boldsymbol{V}^G \cdot \nabla_{\tilde{\boldsymbol{x}}}\,, \tag{38}$$

$$\nabla_{\boldsymbol{x}} = \nabla_{\tilde{\boldsymbol{x}}}\,. \tag{39}$$

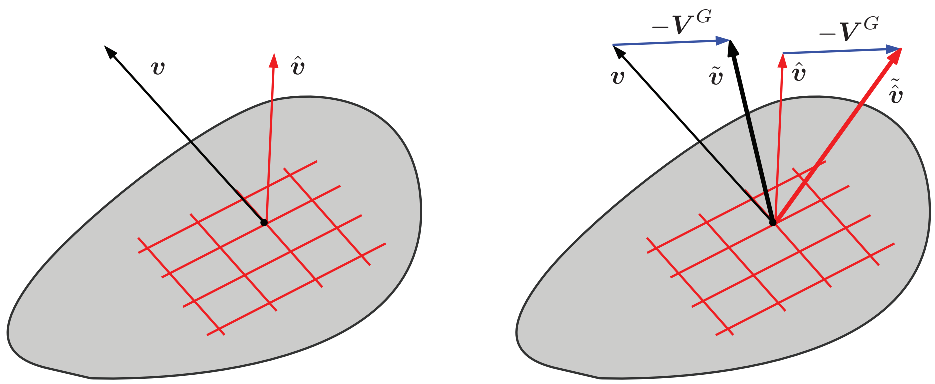

Figure 4. Sketch of a Galilean transformation for a generic ALE mesh. Left: A material domain, and the corresponding mesh, are moving with velocity $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$, respectively. Right: After a Galilean transformation is applied, the material and the mesh are moving with velocities $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}^G$ and $\tilde{\hat{\boldsymbol{v}}} = \hat{\boldsymbol{v}} - \boldsymbol{V}^G$, respectively. The relative velocity of the material with respect to the mesh is an invariant: $\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{v}} - \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G - \hat{\boldsymbol{v}} + \boldsymbol{V}^G = \boldsymbol{v} - \hat{\boldsymbol{v}} = \boldsymbol{c}$.

3.1.2. ALE reference frame In the general ALE case, it is important to observe that $\boldsymbol{\chi}$ is unchanged by a Galilean transformation, since it is the spatial coordinate in the reference frame associated with the initial configuration of the mesh. Hence,

$$\begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{\chi}} \\ \tilde{\boldsymbol{v}} \\ \tilde{\hat{\boldsymbol{v}}} \end{bmatrix} = \begin{bmatrix} t \\ \boldsymbol{\chi} \\ \boldsymbol{v} \\ \hat{\boldsymbol{v}} \end{bmatrix} - \begin{bmatrix} 0 \\ \boldsymbol{0}_{n_d \times 1} \\ \boldsymbol{V}^G \\ \boldsymbol{V}^G \end{bmatrix} . \tag{40}$$

and, in particular $\tilde{\boldsymbol{c}} = \boldsymbol{c}$ (see Fig. 4). In addition, the space/time derivatives are unaffected:

$$\partial_t|_{\boldsymbol{\chi}} = \partial_{\tilde{t}}|_{\tilde{\boldsymbol{\chi}}} , \tag{41}$$

$$\nabla_{\boldsymbol{\chi}} = \nabla_{\tilde{\boldsymbol{\chi}}} . \tag{42}$$

3.2. Galilean transformation of the conservation equations

While the residual of the exact equations are point-wise identically zero, their discrete approximation is not for Galerkin-type methods. In fact, a Galerkin method only enforces that the projection of the residual onto the test space vanishes. Understanding how conservation equations transform is fundamental in evaluating how the variational statement of Galerkin orthogonality is affected by a Galilean change of reference frames. For this purpose, note that the conservative form of the residual can be decomposed as

$$\mathbf{Res}_{\mathbf{U}}(\boldsymbol{Y}) = \begin{bmatrix} \mathsf{Res}^{\rho}(\boldsymbol{Y}) \\ \mathbf{Res}^{\rho \boldsymbol{v}}(\boldsymbol{Y}) \\ \mathsf{Res}^{E}(\boldsymbol{Y}) \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{0}_{1\times n_d} & 0 \\ \boldsymbol{v} & \boldsymbol{I}_{n_d \times n_d} & \boldsymbol{0}_{n_d \times 1} \\ e + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} & \boldsymbol{v}^T & 1 \end{bmatrix} \mathbf{Res}_{\mathbf{A}}(\boldsymbol{Y}) , \tag{43}$$

where

$$\mathbf{Res}_{\mathbf{A}}(\boldsymbol{Y}) = \begin{bmatrix} \mathsf{Res}^{\rho}(\rho; \boldsymbol{c}, \boldsymbol{v}; \boldsymbol{x}, t) \\ \mathsf{Res}_i^{\boldsymbol{v}}(\rho, p; \boldsymbol{c}, \boldsymbol{v}; \boldsymbol{x}, t) \\ \mathsf{Res}^{e}(\rho, e, p; \boldsymbol{c}, \boldsymbol{v}; \boldsymbol{x}, t) \end{bmatrix} = \begin{bmatrix} \partial_t \rho|_{\boldsymbol{\chi}} + \boldsymbol{c}\cdot\nabla_{\boldsymbol{x}}\rho + \rho\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} \\ \rho\, \partial_t \boldsymbol{v}|_{\boldsymbol{\chi}} + \rho(\nabla_{\boldsymbol{x}}\boldsymbol{v})\boldsymbol{c} + \nabla_{\boldsymbol{x}} p - \rho \boldsymbol{g} \\ \rho\, \partial_t e|_{\boldsymbol{\chi}} + \boldsymbol{c}\cdot\nabla_{\boldsymbol{x}} e + p\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} - \rho s \end{bmatrix} \tag{44}$$

is the vector of the residuals associated to the advective form of the equations.

3.2.1. Galilean transformation of the advective residuals Using (41)–(42), it is easily observed that the *advective* mass, momentum, and internal energy residuals are invariant under a Galilean transformation $\mathsf{G}(\cdot)$. That is, recalling $\tilde{\boldsymbol{c}} = \boldsymbol{c}$,

$$\mathsf{G}(\mathsf{Res}^{\rho}(\rho; \boldsymbol{v}, \boldsymbol{c}; \boldsymbol{x}, t)) = \mathsf{Res}^{\rho}(\rho; \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{c}}; \tilde{\boldsymbol{x}}, \tilde{t}) \ , \tag{45}$$

$$\mathsf{G}(\mathbf{Res}^{\boldsymbol{v}}(\rho, p; \boldsymbol{v}, \boldsymbol{c}; \boldsymbol{x}, t)) = \mathbf{Res}^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{c}}; \tilde{\boldsymbol{x}}, \tilde{t}) \ , \tag{46}$$

$$\mathsf{G}(\mathsf{Res}^{e}(\rho, e, p; \boldsymbol{v}, \boldsymbol{c}; \boldsymbol{x}, t)) = \mathsf{Res}^{e}(\rho, e, p; \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{c}}; \tilde{\boldsymbol{x}}, \tilde{t}) \ , \tag{47}$$

or, in more compact form,

$$\boxed{\mathsf{G}(\mathbf{Res}_{\mathsf{A}}(\boldsymbol{Y})) = \mathbf{Res}_{\mathsf{A}}(\tilde{\boldsymbol{Y}}) \ .} \tag{48}$$

Hence, the Galerkin residuals in advective form are invariant under Galilean transformations, independently of the integration quadrature adopted.

3.2.2. Galilean transformation of the conservative residuals A direct computation, using (43) and (48), yields

$$\boxed{\mathsf{G}\left(\mathbf{Res}_{\mathsf{U}}(\boldsymbol{Y})\right) = \mathbf{R}_{\mathsf{G;U}} \ \mathbf{Res}_{\mathsf{U}}(\tilde{\boldsymbol{Y}}) \ ,} \tag{49}$$

with

$$\mathbf{R}_{\mathsf{G;U}} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & 0 \\ \boldsymbol{V}^G & \boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times 1} \\ \frac{1}{2}\boldsymbol{V}^G\cdot\boldsymbol{V}^G & \left(\boldsymbol{V}^G\right)^T & 1 \end{bmatrix} . \tag{50}$$

The matrix $\mathbf{R}_{\mathsf{G;U}}$ has the group property, that is,

$$\mathbf{R}_{\mathsf{G;U}}(\boldsymbol{V}^G + \boldsymbol{W}^G) = \mathbf{R}_{\mathsf{G;U}}(\boldsymbol{V}^G)\mathbf{R}_{\mathsf{G;U}}(\boldsymbol{W}^G) \tag{51}$$

Applying (51) to $\boldsymbol{I} = \mathbf{R}_{\mathsf{G;U}}(\boldsymbol{V}^G + (-\boldsymbol{V}^G))$, it is easy to verify $\mathbf{R}_{\mathsf{G;U}}(-\boldsymbol{V}^G) = \mathbf{R}^{-1}_{\mathsf{G;U}}(\boldsymbol{V}^G)$. An alternative way of deriving (49) is to evaluate the transformation of (14) term by term. This approach eases the understanding of how the overall variational formulation transforms. Starting with the vector $\boldsymbol{U}(\boldsymbol{Y})$,

$$\begin{aligned} \mathsf{G}(\boldsymbol{U}(\boldsymbol{Y})) &= \begin{bmatrix} \rho \\ \rho\left(\tilde{\boldsymbol{v}} + \boldsymbol{V}^G\right) \\ \rho\left(\frac{1}{2}\left(\tilde{\boldsymbol{v}} + \boldsymbol{V}^G\right)\cdot\left(\tilde{\boldsymbol{v}} + \boldsymbol{V}^G\right) + e\right) \end{bmatrix} = \mathbf{R}_{\mathsf{G;U}} \begin{bmatrix} \rho \\ \rho\tilde{\boldsymbol{v}} \\ \rho\left(\frac{1}{2}\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}} + e\right) \end{bmatrix} \\ &= \mathbf{R}_{\mathsf{G;U}} \ \boldsymbol{U}(\tilde{\boldsymbol{Y}}) \ , \end{aligned} \tag{52}$$

Denoting $\tilde{\boldsymbol{U}} = \boldsymbol{U}(\tilde{\boldsymbol{Y}})$, it is easy to realize that

$$\mathbf{R}_{\mathsf{G;U}} = \frac{\partial \boldsymbol{U}}{\partial \tilde{\boldsymbol{U}}} \ . \tag{53}$$

A Galilean transformation applied to $\boldsymbol{G}(\boldsymbol{Y})$ yields

$$\begin{aligned}
\mathsf{G}(\boldsymbol{G}(\boldsymbol{Y})) &= \mathsf{G}\Big(\boldsymbol{U}(\boldsymbol{Y})\boldsymbol{c}^T + \boldsymbol{G}^L(\boldsymbol{Y})\Big) \\
&= \mathbf{R}_{\mathsf{G;U}}\boldsymbol{U}(\tilde{\boldsymbol{Y}})\boldsymbol{c}^T + \begin{bmatrix} \mathbf{0}_{1\times n_d} \\ p\boldsymbol{I}_{n_d\times n_d} \\ p\left(\tilde{\boldsymbol{v}} + \boldsymbol{V}^G\right)^T \end{bmatrix} \\
&= \mathbf{R}_{\mathsf{G;U}}\boldsymbol{U}(\tilde{\boldsymbol{Y}})\boldsymbol{c}^T + \mathbf{R}_{\mathsf{G;U}}\boldsymbol{G}^L(\tilde{\boldsymbol{Y}}) \\
&= \mathbf{R}_{\mathsf{G;U}}\ \boldsymbol{G}(\tilde{\boldsymbol{Y}})\ ,
\end{aligned} \tag{54}$$

and, similarly,

$$\mathsf{G}(\boldsymbol{Z}(\boldsymbol{Y})) = \mathsf{G}\left(-\begin{bmatrix} 0 \\ \rho\boldsymbol{g} \\ \rho(\tilde{\boldsymbol{v}} + \boldsymbol{V}^G)\cdot\boldsymbol{g} + \rho s \end{bmatrix}\right) \tag{55}$$

$$= -\begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & 0 \\ \boldsymbol{V}^G & \boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times 1} \\ \frac{1}{2}\boldsymbol{V}^G\cdot\boldsymbol{V}^G & \left(\boldsymbol{V}^G\right)^T & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \rho\boldsymbol{g} \\ \rho\tilde{\boldsymbol{v}}\cdot\boldsymbol{g} + \rho s \end{bmatrix} \tag{56}$$

$$= \mathbf{R}_{\mathsf{G;U}}\ \boldsymbol{Z}(\tilde{\boldsymbol{Y}})\ . \tag{57}$$

Since $\hat{J}$ is invariant under Galilean transformations, (49) can also be derived as

$$\begin{aligned}
\mathsf{G}(\mathbf{Res}_{\mathsf{U}}(\boldsymbol{Y})) =& \mathsf{G}\Big(\hat{J}^{-1}\ \partial_t|_{\boldsymbol{\chi}}(\hat{J}\ \boldsymbol{U}(\boldsymbol{Y})) + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{G}(\boldsymbol{Y}) + \boldsymbol{Z}(\boldsymbol{Y})\Big) \\
=& \mathbf{R}_{\mathsf{G;U}}\Big(\hat{J}^{-1}\ \partial_{\tilde{t}}|_{\boldsymbol{\chi}}(\hat{J}\ \boldsymbol{U}(\tilde{\boldsymbol{Y}})) + \nabla_{\tilde{\boldsymbol{x}}}\cdot\boldsymbol{G}(\tilde{\boldsymbol{Y}}) + \boldsymbol{Z}(\tilde{\boldsymbol{Y}})\Big)\ .
\end{aligned} \tag{58}$$

3.3. Galilean transformation and variational equations

This section contains the core of the discussion on invariance. Observing that the Neumann flux $\boldsymbol{H}_i$ *must* transform as the vector $\boldsymbol{G}_i$, one has

$$\boldsymbol{H}_i(\boldsymbol{Y}) = \mathbf{R}_{\mathsf{G;U}}\boldsymbol{H}_i(\tilde{\boldsymbol{Y}})\ . \tag{59}$$

As a result, it is not difficult to prove that

$$\boxed{\mathsf{G}\Big(\mathsf{B}(\boldsymbol{W}^h, \tilde{\boldsymbol{Y}}^h) - \mathsf{F}(\boldsymbol{W}^h)\Big) = \mathsf{B}(\mathbf{R}_{\mathsf{G;U}}^T\boldsymbol{W}^h, \tilde{\boldsymbol{Y}}^h) - \mathsf{F}(\mathbf{R}_{\mathsf{G;U}}^T\boldsymbol{W}^h)\ .} \tag{60}$$

In other words, the effect of a Galilean transformation on the Galerkin variational formulation (without stabilization terms) is to transform the test function as

$$\boxed{\tilde{\boldsymbol{W}}^h = \mathbf{R}_{\mathsf{G;U}}^T\boldsymbol{W}^h\ .} \tag{61}$$

Recalling that $\mathbf{R}_{\mathsf{G;U}}^T$ is a constant matrix, it is trivial to realize that *Galilean transformations do not change the test space of the Galerkin projection.* For the sake of completeness, (60) is now verified, using (52), (54), (57), (49), and (59):

$$
\begin{aligned}
\mathsf{G}\Big(\mathsf{B}(\boldsymbol{W}^h,\boldsymbol{Y}^h)\Big) &= \int_{\tilde{\Omega}(\tilde{t}_{n+1})} \boldsymbol{W}^h(\tilde{\boldsymbol{\chi}},\tilde{t}_{n+1}^-)\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{U}(\tilde{\boldsymbol{Y}}^h(\tilde{\boldsymbol{x}},\tilde{t}_{n+1}^-))\; d\tilde{\Omega} \\
&\quad - \int_{\tilde{\Omega}(\tilde{t}_n)} \boldsymbol{W}^h(\tilde{\boldsymbol{\chi}},\tilde{t}_n^+)\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{U}(\tilde{\boldsymbol{Y}}^h(\tilde{\boldsymbol{x}},\tilde{t}_n^-))\;\; d\tilde{\Omega} \\
&\quad - \int_{Q_n} \Big(\boldsymbol{W}^h_{,\tilde{t}}\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{U}(\tilde{\boldsymbol{Y}}^h) + \boldsymbol{W}^h_{,\tilde{x}_i}\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{G}_i(\tilde{\boldsymbol{Y}}^h)\Big)\; d\tilde{Q} \\
&\quad + \int_{Q_n} \boldsymbol{W}^h\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{Z}(\tilde{\boldsymbol{Y}}^h)\; d\tilde{Q} + \int_{P_n^g} \boldsymbol{W}^h\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{G}_i(\tilde{\boldsymbol{Y}}^h) n_i d\tilde{P} \\
&= \int_{\tilde{\Omega}(\tilde{t}_{n+1})} \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h(\tilde{\boldsymbol{\chi}},\tilde{t}_{n+1}^-)\cdot \boldsymbol{U}(\tilde{\boldsymbol{Y}}^h(\tilde{\boldsymbol{x}},\tilde{t}_{n+1}^-))\; d\tilde{\Omega} \\
&\quad - \int_{\tilde{\Omega}(\tilde{t}_n)} \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h(\tilde{\boldsymbol{\chi}},\tilde{t}_n^+)\cdot \boldsymbol{U}(\tilde{\boldsymbol{Y}}^h(\tilde{\boldsymbol{x}},\tilde{t}_n^-))\;\; d\tilde{\Omega} \\
&\quad - \int_{Q_n} \Big(\mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h_{,\tilde{t}}\cdot \boldsymbol{U}(\tilde{\boldsymbol{Y}}^h) + \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h_{,\tilde{x}_i}\cdot \boldsymbol{G}_i(\tilde{\boldsymbol{Y}}^h)\Big)\; d\tilde{Q} \\
&\quad + \int_{Q_n} \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h\cdot \boldsymbol{Z}(\tilde{\boldsymbol{Y}}^h)\; d\tilde{Q} + \int_{P_n^g} \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h\cdot \boldsymbol{G}_i(\tilde{\boldsymbol{Y}}^h) n_i d\tilde{P} \\
&= \mathsf{B}(\mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h,\tilde{\boldsymbol{Y}}^h)\;,
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
\mathsf{G}\Big(\mathsf{F}(\boldsymbol{W}^h)\Big) &= -\int_{\tilde{P}_n^h} \boldsymbol{W}^h\cdot \mathbf{R}_{\mathsf{G;U}}\boldsymbol{H}_i(\tilde{\boldsymbol{Y}}) n_i\; d\tilde{P} \\
&= -\int_{\tilde{P}_n^h} \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h\cdot \boldsymbol{H}_i(\tilde{\boldsymbol{Y}}) n_i\; d\tilde{P} \\
&= \mathsf{F}(\mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{W}^h)\;.
\end{aligned} \tag{63}
$$

SUPG methods provide a perturbation to the Galerkin test function which enhances the stability of the overall variational formulation in the discrete case. Using (49), it is straightforward to derive

$$
\begin{aligned}
\mathsf{G}\Big(\mathsf{SUPG}(\boldsymbol{P}^h(\boldsymbol{W}^h),\boldsymbol{Y}^h)\Big) &= -\sum_{e=1}^{(n_{el})_n} \int_{\tilde{Q}_n^e} \boldsymbol{P}^h(\boldsymbol{W}^h)\cdot \mathbf{R}_{\mathsf{G;U}}\mathbf{Res}_{\mathsf{U}}(\tilde{\boldsymbol{Y}}^h)\; d\tilde{Q} \\
&= -\sum_{e=1}^{(n_{el})_n} \int_{\tilde{Q}_n^e} \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{P}^h(\boldsymbol{W}^h)\cdot \mathbf{Res}_{\mathsf{U}}(\tilde{\boldsymbol{Y}}^h)\; d\tilde{Q} \\
&= \mathsf{SUPG}(\mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{P}^h(\boldsymbol{W}^h),\tilde{\boldsymbol{Y}}^h)\;,
\end{aligned} \tag{64}
$$

so that, similarly to equation (61) for $\boldsymbol{W}^h$,

$$
\boxed{\tilde{\boldsymbol{P}}^h = \mathbf{R}^T_{\mathsf{G;U}}\boldsymbol{P}^h\;.} \tag{65}
$$

Using the duality pairing notation,

$$\langle \mathsf{Res}_{\mathsf{U}}(\boldsymbol{Y}^h), \boldsymbol{W}^h + \boldsymbol{P}^h \rangle = \mathsf{B}(\boldsymbol{W}^h, \boldsymbol{Y}^h) + \mathsf{SUPG}(\boldsymbol{P}^h(\boldsymbol{W}^h), \tilde{\boldsymbol{Y}}^h) - \mathsf{F}(\boldsymbol{W}^h) \,, \tag{66}$$

the projection of the residual operator onto the Petrov/Galerkin SUPG space transforms as

$$\boxed{\mathsf{G}\Big(\langle \mathsf{Res}_{\mathsf{U}}(\boldsymbol{Y}^h), \boldsymbol{W}^h + \boldsymbol{P}^h \rangle\Big) = \langle \mathsf{Res}_{\mathsf{U}}(\tilde{\boldsymbol{Y}}^h), \mathsf{R}^T_{\mathsf{G};\mathsf{U}}(\boldsymbol{W}^h + \boldsymbol{P}^h) \rangle \,.} \tag{67}$$

At this point it is important to observe that the Galerkin test function $\boldsymbol{W}^h(\boldsymbol{x}, t)$ is translated in space by a Galilean transformation, but its relative shape remains unchanged. This can be quantitatively expressed by the relation

$$\boldsymbol{W}^h(\boldsymbol{x}, t) = \boldsymbol{W}^h(\tilde{\boldsymbol{x}} + \boldsymbol{V}^G \tilde{t}, \tilde{t}) \,, \text{ or, } \boldsymbol{W}^h(\tilde{\boldsymbol{\chi}}, t) = \boldsymbol{W}^h(\tilde{\boldsymbol{\chi}}, \tilde{t}) \,. \tag{68}$$

It is now natural to expect that a similar relationship should hold also for the SUPG perturbation to the test function, $\boldsymbol{P}^h$. If this is not the case, one would have that the stability property of the method may be affected by the observer's reference frame. Hence, we must impose that the shape of $\boldsymbol{P}^h$ is invariant under a Galilean transformation, that is

$$\boxed{\boldsymbol{P}^h(\boldsymbol{x}, t) = \boldsymbol{P}^h(\tilde{\boldsymbol{x}} + \boldsymbol{V}^G \tilde{t}, \tilde{t})} \,, \text{ or, } \boxed{\boldsymbol{P}^h(\tilde{\boldsymbol{\chi}}, t) = \boldsymbol{P}^h(\tilde{\boldsymbol{\chi}}, \tilde{t})} \,. \tag{69}$$

This is a necessary condition for Galilean frame invariance of the SUPG variational formulation, and, ultimately, the numerical solution. Relation (69) is expressing the core of the entire discussion presented so far: Once it is understood how a Galilean transformation affects the test function $\boldsymbol{W}^h$, the same rules should hold for $\boldsymbol{P}^h$, so that the test function space is preserved invariant under Galilean transformations.

3.4. Design of Galilean invariant SUPG operators

The following discussion evaluates invariance properties of specific forms of $\boldsymbol{P}^h(\boldsymbol{W}^h)$, and a detailed expression needs to be provided. It is standard practice in SUPG formulations [5, 12, 19] to define

$$\boldsymbol{P}^h(\boldsymbol{W}^h) = -\,\boldsymbol{\tau}^T \left(\hat{\boldsymbol{A}}_0^T \, \partial_t|_{\boldsymbol{\chi}} - \hat{\boldsymbol{A}}_i^T \, \partial_{x_i} \right) \boldsymbol{W}_h \,, \tag{70}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\Delta t, h_e, \hat{\boldsymbol{A}}_0, \hat{\boldsymbol{A}}_i, \hat{\boldsymbol{C}}, \dots) \,, \tag{71}$$

where Δt is the time increment, and h_e is the eth element mesh scale. The previous expression for $\boldsymbol{\tau}$, which includes its functional dependence on the parameters and various terms in the formulation, is sufficient to fully understand the issues under investigation. The matrices $\hat{\boldsymbol{A}}_0$, $\hat{\boldsymbol{A}}_i$, $\hat{\boldsymbol{C}}$, represent generalizations of the matrices $\boldsymbol{A}_0$, $\boldsymbol{A}_i$, $\boldsymbol{C}$. Specifically,

$$\hat{\boldsymbol{A}}_0 = \mathbf{T}_{\mathsf{P}}(\boldsymbol{Y}) \ \boldsymbol{A}_0 \,, \tag{72}$$

$$\hat{\boldsymbol{A}}_i = \mathbf{T}_{\mathsf{P}}(\boldsymbol{Y}) \ \boldsymbol{A}_i \,, \tag{73}$$

$$\hat{\boldsymbol{C}} = \mathbf{T}_{\mathsf{P}}(\boldsymbol{Y}) \ \boldsymbol{C} \,. \tag{74}$$

Condition (69) is equivalent to saying that $\hat{\boldsymbol{A}}_0\boldsymbol{\tau}$ and $\hat{\boldsymbol{A}}_i\boldsymbol{\tau}$ must not change under a Galilean transformation. This poses some limitations on the form that such terms can take. As shown in [15, 16], taking $\boldsymbol{\tau}$ proportional to $\boldsymbol{A}_0^{-1}$ and $\mathsf{T_P} = \boldsymbol{I}$ in the definitions of $\hat{\boldsymbol{A}}_0$, $\hat{\boldsymbol{A}}_i$, and $\hat{\boldsymbol{C}}$, leads to products $\boldsymbol{A}_0\boldsymbol{\tau}$ and $\boldsymbol{A}_i\boldsymbol{\tau}$ which explicitly contain the fluid velocity. Because the fluid velocity changes with a change of reference frames, the perturbation $\boldsymbol{P}^h$ obtained with this procedure is *observer dependent*. In fact, denoting by $\kappa = \boldsymbol{v}\cdot\boldsymbol{v}/2$ the specific kinetic energy, and by $H = e + \kappa + p/\rho$ the specific total enthalpy,

$$\boldsymbol{A}_i\boldsymbol{A}_0^{-1} = \left[\begin{array}{c|c|c|c|c} c_i - v_i & \delta_{1i} & \delta_{2i} & \delta_{3i} & 0 \\ \hline \frac{\kappa - e - \rho e_{,\rho}}{\rho e_{,p}}\delta_{1i} - v_1 v_i & c_i - v_i - \frac{v_1\delta_{1i}}{\rho e_{,p}} & 0 & 0 & \delta_{1i} \\ \hline \frac{\kappa - e - \rho e_{,\rho}}{\rho e_{,p}}\delta_{2i} - v_2 v_i & 0 & c_i - v_i - \frac{v_2\delta_{2i}}{\rho e_{,p}} & 0 & \delta_{2i} \\ \hline \frac{\kappa - e - \rho e_{,\rho}}{\rho e_{,p}}\delta_{3i} - v_3 v_i & 0 & 0 & c_i - v_i - \frac{v_3\delta_{3i}}{\rho e_{,p}} & \delta_{3i} \\ \hline \left(\frac{\kappa - e}{\rho e_{,p}} - H\right) v_i & H\delta_{1i} - \frac{v_1 v_i}{\rho e_{,p}} & H\delta_{2i} - \frac{v_2 v_i}{\rho e_{,p}} & H\delta_{3i} - \frac{v_3 v_i}{\rho e_{,p}} & c_i + \frac{v_i}{\rho e_{,p}} \end{array}\right] \tag{75}$$

is *not* Galilean invariant. It was also shown in [15, 16] that using the matrix Jacobians relative to the advective form of the residuals (44) yields Galilean invariant SUPG test function perturbations, in the case of a solution vector $\boldsymbol{Y}$ given by $[\rho,\ \boldsymbol{v}^T,\ p]^T$, $[\rho,\ \boldsymbol{v}^T,\ e]^T$, or $[e,\ \boldsymbol{v}^T,\ p]^T$. Specifically, in the case of $\boldsymbol{Y} = [\rho,\ \boldsymbol{v}^T,\ p]^T$, it is straightforward to realize that, if

$$\mathsf{T_P} = \begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & 0 \\ -\boldsymbol{v} & \boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times 1} \\ \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} - e & -\boldsymbol{v}^T & 1 \end{bmatrix}, \tag{76}$$

then

$$\mathsf{Res_A}(\boldsymbol{Y}) = \hat{\boldsymbol{A}}_0\ \partial_t|_{\boldsymbol{\chi}}\boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y})\ \partial_{x_i}\boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y})\ \boldsymbol{Y}\ . \tag{77}$$

Hence, the advective Jacobians $\hat{\boldsymbol{A}}_0$, $\hat{\boldsymbol{A}}_i$, and $\hat{\boldsymbol{C}}$ can be expressed in terms of the conservative Jacobians $\boldsymbol{A}_0$, $\boldsymbol{A}_i$, and $\boldsymbol{C}$, by means of an appropriate transformation matrix $\mathsf{T_P}$. Note also that the structure of the hyperbolic operators governing the flow equations is already captured by the advective Jacobians. Denoting $e_{,p} = \partial_p e|_\rho$ and $e_{,\rho} = \partial_\rho e|_p$, one has that

$$\begin{aligned} \hat{\boldsymbol{A}}_0 &= \mathsf{T_P}\boldsymbol{A}_0 \\ &= \begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & 0 \\ -\boldsymbol{v} & \boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times 1} \\ \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} - e & -\boldsymbol{v}^T & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & 0 \\ \boldsymbol{v} & \rho\boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times 1} \\ \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + e + \rho e_{,\rho} & -\boldsymbol{v}^T & \rho e_{,p} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathbf{0}_{1\times n_d} & 0 \\ \mathbf{0}_{n_d\times 1} & \rho\boldsymbol{I}_{n_d\times n_d} & \mathbf{0}_{n_d\times 1} \\ \rho e_{,\rho} & \mathbf{0}_{1\times n_d} & \rho e_{,p} \end{bmatrix} \end{aligned} \tag{78}$$

is Galilean invariant. Analogously, for $i = 1, 2, 3$, $\boldsymbol{A}_i$ can be decomposed as $\boldsymbol{A}_i = c_i\boldsymbol{A}_0 + \boldsymbol{A}_i^{L;I} + \boldsymbol{A}_i^{L;II}$, with,

$$\boldsymbol{A}_i^{L;I} = \left[\begin{array}{c|c|c|c|c} \mathbf{0}_{(n_d+2)\times 1} & \boldsymbol{U}\delta_{1i} & \boldsymbol{U}\delta_{2i} & \boldsymbol{U}\delta_{3i} & \mathbf{0}_{(n_d+2)\times 1} \end{array}\right], \tag{79}$$

$$\boldsymbol{A}_i^{L;II} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \delta_{1i} \\ \hline 0 & 0 & 0 & 0 & \delta_{2i} \\ \hline 0 & 0 & 0 & 0 & \delta_{3i} \\ \hline 0 & p\,\delta_{1i} & p\,\delta_{2i} & p\,\delta_{3i} & v_i \end{array}\right], \tag{80}$$

where δ_{ij} is the Kronecker tensor. With a little algebra,

$$\mathsf{T}_{\mathsf{P}}\boldsymbol{A}_i^{L;I} = \left[\begin{array}{c|c|c|c|c} 0 & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{81}$$

$$\mathsf{T}_{\mathsf{P}}\boldsymbol{A}_i^{L;II} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \delta_{1i} \\ \hline 0 & 0 & 0 & 0 & \delta_{2i} \\ \hline 0 & 0 & 0 & 0 & \delta_{3i} \\ \hline 0 & p\,\delta_{1i} & p\,\delta_{2i} & p\,\delta_{3i} & 0 \end{array}\right], \tag{82}$$

so that

$$\hat{\boldsymbol{A}}_i = \left[\begin{array}{c|c|c|c|c} c_i & \rho\;\delta_{1i} & \rho\;\delta_{2i} & \rho\;\delta_{3i} & 0 \\ \hline 0 & \rho c_i & 0 & 0 & \delta_{1i} \\ \hline 0 & 0 & \rho c_i & 0 & \delta_{2i} \\ \hline 0 & 0 & 0 & \rho c_i & \delta_{3i} \\ \hline \rho c_i e_{,\rho} & p\,\delta_{1i} & p\,\delta_{2i} & p\,\delta_{3i} & \rho c_i e_{,p} \end{array}\right] \tag{83}$$

is Galilean invariant. Similar results hold for the solution variables $[\rho,\ \boldsymbol{v}^T,\ e]^T$ and $[e,\ \boldsymbol{v}^T,\ p]^T$, as shown in [15, 16]. In particular, the definition

$$\boldsymbol{\tau} = \alpha\frac{\Delta t}{2}\hat{\boldsymbol{A}}_0^{-1} = \alpha\frac{\Delta t}{2}\boldsymbol{A}_0^{-1}\mathsf{T}_{\mathsf{P}}^{-1} \tag{84}$$

was used in [17, 18] to derive appropriate stabilization operators in the Lagrangian case (α is a scalar factor, which does not depend on the fluid velocity). In fact, $\hat{\boldsymbol{A}}_0\boldsymbol{\tau}$ and $\hat{\boldsymbol{A}}_i\boldsymbol{\tau}$ are products of Galilean invariant matrices, thus invariant.

In the case of conservation variables ($\boldsymbol{Y} = \boldsymbol{U}$) or entropy variables [19], it is difficult (although in principle not impossible) to derive expressions for the matrices $\hat{\boldsymbol{A}}_0$ and $\hat{\boldsymbol{A}}_i$ which preserve Galilean invariance of $\boldsymbol{P}^h$. This is due to the strong nonlinearities in the expression for $\boldsymbol{G}(\boldsymbol{U})$ [15, 16].

Finally, note that Galilean invariance properties are intrinsically *local*. For example, a non-invariant SUPG operator *will not* affect the global balance of mass, momentum or total energy. As is well known, this is due to the fact the $\boldsymbol{P}^h$ is a function of gradients in space and time of $\boldsymbol{W}^h$ (see definition (70)). Hence, when the unit constant is used to test the variational formulation (as customary in the proof of global conservation properties), the SUPG term vanishes. This means that SUPG operators lacking Galilean invariance properties can change local conservation budgets, while leaving global conservation budgets unchanged. Hence, a test of frame invariance of global energy budgets is not sufficient to prove (local) invariance of the formulation. The locality aspect of Galilean invariance can clearly spur discussion on the use of non-invariant SUPG operators in simulations of compressible turbulence, in which case the local energy budgets underlying the turbulent kinetic energy cascade may be affected.

4. Concluding remarks

This article presents a generalization and further elaboration of previous work by the first author in the context of Galilean invariance of stabilization operators [15, 16]. The proposed analysis confirms the previous findings and broadens the perspective on how to consistently derive invariant stabilizing operators. The problem of how to generate invariant stabilized discretizations for the case of conservation and entropy variables is still open, and is object of ongoing research.

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